

# PRECOMPACTNESS OF RADIAL EXTREMIZING SEQUENCES FOR A $k$ -PLANE TRANSFORM INEQUALITY.

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ABSTRACT. Let  $d \geq 2$  and  $1 \leq k \leq d - 1$ . It is known that the  $k$ -plane transform satisfy some  $L^p \rightarrow L^q$  dilation-invariant inequality, for which radial extremizers exist. We show that in the endpoint case, radial extremizing sequences are relatively compact modulo the group of dilations.

## 1. INTRODUCTION AND STATEMENT OF RESULTS

Let us choose  $d \geq 2$ ,  $1 \leq k \leq d - 1$  and denote by  $\mathcal{G}_k$  the set of all  $k$ -planes in  $\mathbb{R}^d$ , that means affine subspaces in  $\mathbb{R}^d$  with dimension  $k$ . We define the  $k$ -plane transform of a continuous function with compact support  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  as

$$\mathcal{R}_k f(\Pi) = \int_{\Pi} f d\lambda_{\Pi}$$

where  $\Pi \in \mathcal{G}_k$  and the measure  $\lambda_{\Pi}$  is the surface Lebesgue measure on  $\Pi$ . It is well known since the works of Oberlin and Stein [9], Drury [6] and Christ [1] that  $\mathcal{R}_k$  can be extended from  $L^{\frac{d+1}{k+1}}(\mathbb{R}^d)$  to  $L^{d+1}(\mathcal{G}_k)$  for a certain measure on  $\mathcal{G}_k$  that was defined in the introduction of [5].

The corresponding  $L^{\frac{d+1}{k+1}}(\mathbb{R}^d)$  to  $L^{d+1}(\mathcal{G}_k, \sigma_k)$  inequality is

$$\|\mathcal{R}_k f\|_{L^{d+1}(\mathcal{G}_k, \sigma_k)} \leq A(k, d) \|f\|_{L^{\frac{d+1}{k+1}}(\mathbb{R}^d)}, \quad (1.1)$$

for a certain constant  $A(k, d)$  chosen to be optimal. The value of this constant is known, and so are some extremizers. However, the question of uniqueness is still open for  $k \neq d - 1$ . Let us recall the following theorem from [5]:

**Theorem 1.** (i) *There exist radial, nonincreasing extremizers for (1.1). Moreover, any extremizing sequence of nonincreasing, radial functions is relatively compact - modulo the group of dilations.*

(ii) *Some extremizers for (1.1) are given by*

$$h(x) = \left[ \frac{C}{1 + \|Lx\|^2} \right]^{\frac{k+1}{2}}$$

where  $L$  is any invertible affine map on  $\mathbb{R}^d$ , and  $C$  is a constant.

(iii) The best constant  $A(k, d)$  is equal to

$$\frac{\|\mathcal{R}_k h\|_{d+1}}{\|h\|_{\frac{d+1}{k+1}}} = \left[ 2^{k-d} \frac{|S^k|^d}{|S^d|^k} \right]^{\frac{1}{d+1}},$$

where  $|S^{i-1}|$  denotes the Lebesgue surface measure of the euclidean sphere in  $\mathbb{R}^i$ .

Part (i) of Theorem 1 is a pretty restricted result and it is natural to ask for more generality. Let us recall the concept of extremizing sequence:

**Definition 1.1.** A sequence of  $L^{\frac{d+1}{k+1}}$  functions  $f_n$  is said to be an extremizing sequence for (1.1) if for all  $n$ ,  $\|f_n\|_{\frac{d+1}{k+1}} = 1$  and  $\|\mathcal{R}_k f_n\|_{d+1} \rightarrow A(k, d)$  as  $n \rightarrow \infty$ .

In [3] then [4], Christ proved that for  $k = d - 1$ , any extremizing sequence  $f_n$  admits a subsequence that converges modulo the group of affine maps. This means that there exists a sequence of affine maps  $L_n$  with determinant equal to 1 such that  $f_n \circ L_n$  converges in  $L^{\frac{d+1}{d}}$ . We are concerned here by extremizing sequences of radial functions, *not necessary nonincreasing*. It is of course harder to prove that they are relatively compact when they are not supposed to be nonincreasing. Indeed, sequences of  $L^{\frac{d+1}{k+1}}$  nonincreasing, radial functions converge almost everywhere, modulo a subsequence, as a consequence of Helly's principle.

Let us fix  $k, d$  and call

$$p = \frac{d+1}{k+1}, \quad q = d+1.$$

As carefully explained in [5], restraining the inequality (1.1) to radial functions is equivalent to consider the simpler one-dimensional operator  $\mathcal{T}$ , formally defined as

$$\mathcal{T}f(r) = \int_0^\infty f(\sqrt{r^2 + s^2}) s^{k-1} ds = \int_r^\infty f(u) (u^2 - r^2)^{k/2-1} u du.$$

This operator maps  $L^p((0, \infty), r^{d-1} dr)$  to  $L^q((0, \infty), r^{d-k-1} dr)$ . The corresponding inequality is

$$\|\mathcal{T}f\|_{L^q((0, \infty), r^{d-k-1} dr)} \leq B(k, d) \|f\|_{L^p((0, \infty), r^{d-1} dr)} \quad (1.2)$$

for an optimal constant  $B(k, d)$  that has been computed in [5]. Note that the inequality (1.2) is dilation-invariant: that was the main difficulty in the proof of part (i) of Theorem 1. Although  $\mathcal{T}$  enjoys a much simpler behavior than the  $k$ -plane transform, the reader should keep in mind its connection to  $\mathcal{R}_k$ . The  $k$ -plane transform has a geometric origin that provides a nice and simple geometric interpretation of the action of  $\mathcal{T}$ . This has guided our intuition for most of the lemma (and the associated proofs) that will follow.

**Main result and outline of the proof.** Our goal is to prove the following theorem:

**Theorem 2.** *Let  $1 \leq k \leq d-1$  and  $f_n$  be an extremizing sequence for (1.2). Then  $f_n$  is relatively compact in  $L^p(r^{d-1}dr)$  modulo the group of dilations, that is, there exists a sequence  $\lambda_n \in \mathbb{R}$  such that*

$$r \mapsto \lambda_n^{d/p} f_n(\lambda_n r)$$

*converges strongly, modulo a subsequence.*

Since Christ thoroughly studied the case  $k = d-1$  in [3] and [4] we will restrict our attention to the case  $d \geq 3$ .

The strategy is outlined as follows. Let  $f_n$  be an extremizing sequence.  $f_n$  admits a weak limit, called  $f$ , and our goal is to prove that  $f$  is actually a strong limit. We thus would like to use a compact operator to transform this weak limit into a strong limit. Of course, here the natural operator to use is  $\mathcal{T}$ . We are missing an assumption:  $\mathcal{T}$  as defined earlier is not compact. But it remains a kernel operator, and being so it is not far from being compact. Its kernel

$$k(r, s) = \mathbb{1}_{s \geq r} (s^2 - r^2)^{k/2-1} s$$

admits essentially two singularities, one for  $r = s$  and  $k = 1$ , and another one for large values of  $s$ . As a consequence, we define a truncated operator,

$$\begin{aligned} \mathcal{T}_R : L^\infty([0, R]) &\longrightarrow L^q([0, R]) \\ f &\longmapsto \mathcal{T} \mathbb{1}_{[0, R]} f = \mathcal{T} f. \end{aligned}$$

In this setting, since we artificially erased the singularities, for any  $0 < R < \infty$ ,  $\mathcal{T}_R$  is compact – see the appendix.

Write now  $f_n = g_n^m + \varepsilon_n^m$  with  $g_n^m := \mathbb{1}_{[0, m]} \mathbb{1}_{\{f_n \leq m\}} f_n$ . We want to take the limit of this equality as  $n, m$  tends to infinity. Let  $\mathcal{T}$  acts on this equality:

$$\mathcal{T} f_n = \mathcal{T}_m g_n^m + \mathcal{T} \varepsilon_n^m.$$

It would be convenient to be able to take the limit  $n \rightarrow \infty$  then  $m \rightarrow \infty$  for  $\mathcal{T}_m g_n^m$  and  $m \rightarrow \infty$  then  $n \rightarrow \infty$  for  $\mathcal{T} \varepsilon_n^m$ . This is possible if we can prove the two following facts:

- (i)  $h^m$  converges in  $L^q$  as  $m \rightarrow \infty$ , where  $h^m$  is the strong limit of  $\mathcal{T}_m g_n^m$ ;
- (ii)  $\varepsilon_n^m$  is uniformly small in  $n$  as  $m$  tends to infinity.

(i) is actually pretty easy, it follows from the structure of  $\mathcal{T}$ . It will be proved in section 4. (ii) is actually much harder and will require two uniform bounds on  $f_n$ . These bounds can be obtained only after an adapted rescaling of the sequence  $(f_n)$ . The proof of (ii) is separated into to parts:

- (1) In section 2, we rescale our extremizing sequence to make it converge to a non-zero function, in the weak sense. Some arguments related to (i) in Theorem 1

will lead to the impossibility of the sequence to loose some weight as a Dirac mass, expressed as a first uniform bound in  $n$ :

$$\lim_{m \rightarrow \infty} \int_{|f_n| \geq m} |f_n|^p = 0.$$

- (2) In section 3 we use the concentration-compactness principle of Lions [8] to prove a strong, and uniform in  $n$ , localization result on the extremizing sequence, expressed as

$$\lim_{m \rightarrow \infty} \int_{r \geq m} |f_n|^p = 0.$$

### Notations.

- Let  $A$  and  $B$  be positive functions and  $P$  be some statement. We will say that  $P$  implies that  $A \lesssim B$  when there exists a universal constant  $C$ , which depends only on  $k$  and  $d$ , such that  $P$  implies that  $A \leq CB$ .  $A \gtrsim B$  will be the convert and  $A \sim B$  will be used when  $A \lesssim B$  and  $B \lesssim A$ .
- $\|f\|_p$  denotes the  $L^p$ -norm of  $f$ , with respect to a contextual measure.
- For  $f$  measurable, we denote  $f^*$  the radial, nonincreasing rearrangement of  $f$ , with respect to a contextual measure. It is the only radial nonincreasing function whose level sets have the same measure as the level sets of  $f$ , see [7] for a more complete introduction.
- For a measurable set  $E$  and a contextual measure we will denote by  $|E|$  its measure.
- For two sets  $E, F \subset \mathbb{R}$ ,  $d(E, F)$  denotes the standard distance between  $E$  and  $F$ . If  $R > 0$ ,  $d(R, E)$  denotes the distance between  $[0, R]$  and  $E$ .

Our purpose is to prove Theorem 2 for a subsequence of an extremizing sequence  $f_n$ . To simplify the notations, we will still call  $f_n$  any subsequence that is extracted from  $f_n$ .

## 2. A CONCENTRATION-COMPACTNESS RESULT

We first want to prove the following:

**Lemma 2.1.** *Let  $f_n$  be an extremizing sequence for (1.2). There exists  $R_0$  such that the following is satisfied. For all  $n$ , there exist a set  $E_n \subset \mathbb{R}$ , and  $\lambda_n \in \mathbb{R}$ , such that if*

$$g_n : r \mapsto \lambda_n^{d/p} f_n(\lambda_n r),$$

*then:*

- (1)  $|E_n| \sim 1$ ;
- (2)  $g_n \geq \mathbb{1}_{E_n}$ ;
- (3)  $E_n \subset [0, R_0]$ .

This lemma is somehow an improvement of the concentration-compactness lemma that we proved in [5]. There are essentially three things to identify:  $\lambda_n$ ,  $E_n$ , and  $R_0$ .  $\lambda_n$  and  $E_n$  will be found by similar method as in the proof of Lemma 3.4 in [5].  $R_0$  is the result of refined inequalities concerning  $\mathcal{T}$ .

*Proof.* We start by using the proof of the concentration-compactness Lemma 3.4 in [5]. It tells us that there exists  $\lambda_n$  such that the sequence

$$\rho_n := \lambda_n^{d/p} f_n^*(\lambda_n \cdot)$$

satisfies  $\rho_n \geq \mathbb{1}_{[0, R_n]}$  with  $R_n \sim 1$ . Let us call  $g_n := \lambda_n^{d/p} f_n(\lambda_n \cdot)$ . Then  $g_n^* = \rho_n$ , which implies that there exists a set  $\mathcal{E}_n$  with measure of order 1 such that  $g_n \geq \mathbb{1}_{\mathcal{E}_n}$ . We will call  $e = |\mathcal{E}_n| > 0$ . This number  $e$  can be chosen independent on  $n$  since a lower bound on  $|\mathcal{E}_n|$  exists.

All that remain to be shown is that the sets  $\mathcal{E}_n$  have most of their weight uniformly close from 0. Let us chose  $R > 0$  and call  $F_n = \mathcal{E}_n \setminus [0, R]$ ,  $\delta_n(R) = |F_n|$ . We want to show that if  $\liminf_n \delta_n(R)$  is bounded from below when  $R \rightarrow \infty$ , then  $g_n$  cannot be an extremizing sequence. We then consider  $h_n = g_n - \mathbb{1}_{F_n}$ . We have:

$$\|h_n\|_p^p = \int_0^\infty [g_n - \mathbb{1}_{F_n}]^p \leq \int_0^\infty g_n^p - \mathbb{1}_{F_n} \leq 1 - \delta_n(R).$$

On the other hand,

$$\|\mathcal{T}h_n\|_q \geq \|\mathcal{T}g_n\|_q - \|\mathcal{T}\mathbb{1}_{F_n}\|_q.$$

Thus we need to give upper bounds on  $\|\mathcal{T}\mathbb{1}_{F_n}\|_q$ . Here because of the distinction between  $k = 1$  and  $k \geq 2$  we have to prove an estimate for each case. Let us start with  $k \geq 2$ .

**Lemma 2.2.** *Let  $k \geq 2$ ,  $R > 0$  and  $F \subset \mathbb{R}$  such that  $|F| = \delta$  and  $[0, R] \cap F = \emptyset$ . Then*

$$\|\mathcal{T}\mathbb{1}_F\|_q \leq 2\delta R^{\frac{d(k-d)}{d+1}}.$$

*Proof.* We start by a pointwise estimate: let  $r > 0$ , we want upper bounds on  $\mathcal{T}\mathbb{1}_F(r)$ .

$$\begin{aligned} \mathcal{T}\mathbb{1}_F(r) &= \int_{u \geq \max(r, R)} \mathbb{1}_F(u) (u^2 - r^2)^{k/2-1} u du \\ &= \int_{u \geq \max(r, R)} \mathbb{1}_F(u) u^{d-1} (u^2 - r^2)^{k/2-1} u^{2-d} du \\ &\leq \max(r, R)^{k-d} \delta. \end{aligned}$$

Here we used  $k \geq 2$ . Thus

$$\begin{aligned} \|\mathcal{T}\mathbb{1}_F\|_q^q &\leq \int_{r \leq R} [R^{k-d} \delta]^q r^{d-k-1} dr + \int_{r \geq R} (r^{k-d} \delta)^q r^{d-k-1} dr \\ &\leq 2\delta^q R^{d(k-d)}. \end{aligned}$$

This immediately leads to

$$\|\mathcal{T}\mathbf{1}_F\|_q \leq 2\delta R^{\frac{d(k-d)}{d+1}} = 2\delta R^{-d/p}.$$

□

A similar lemma for  $k = 1$  is the following:

**Lemma 2.3.** *Let  $k = 1$ ,  $R \geq 1$ ,  $F$  a measurable set with  $|F| \sim 1$  such that  $[0, R] \cap F = \emptyset$ . Then*

$$\|\mathcal{T}\mathbf{1}_F\|_q \lesssim R^{-1/q}.$$

The proof is essentially in three steps: first we prove it when  $F$  is an interval, then when  $F$  is a countable union of intervals, and at last when  $F$  is any measurable set.

*Proof.* Let  $\rho \geq R$  and  $F$  be the set  $(\rho, \rho + \delta)$ . Then

$$\delta \rho^{d-1} \sim |F| \sim 1.$$

Moreover,

$$\mathcal{T}\mathbf{1}_F(r) \leq \sqrt{(\rho + \delta)^2 - \rho^2} \sim \sqrt{\delta \rho}.$$

Let us look at  $\|\mathcal{T}\mathbf{1}_F\|_q^q$ ; choose  $1 \geq \varepsilon \geq \delta$ . We can cut the integral into two parts:

$$\begin{aligned} \|\mathcal{T}\mathbf{1}_F\|_q^q &= \int_0^{\rho-\varepsilon} |\mathcal{T}\mathbf{1}_F|^q + \int_{\rho-\varepsilon}^{\rho+\delta} |\mathcal{T}\mathbf{1}_F|^q \\ &\lesssim \left( \int_0^{\rho-\varepsilon} |\mathcal{T}\mathbf{1}_F|^q \right) + (\delta \rho)^{q/2} \rho^{d-2} \varepsilon. \end{aligned}$$

Now we give estimates on

$$\int_0^{\rho-\varepsilon} |\mathcal{T}\mathbf{1}_F|^q.$$

On  $(0, \rho - \varepsilon)$ ,

$$\begin{aligned} \mathcal{T}\mathbf{1}_F(r) &\leq \mathcal{T}\mathbf{1}_F(\rho - \varepsilon) \\ &\leq \sqrt{(\rho + \delta)^2 - (\rho - \varepsilon)^2} - \sqrt{\rho^2 - (\rho - \varepsilon)^2} \\ &\leq \sqrt{2\rho\delta + \delta^2 + 2\rho\varepsilon - \varepsilon^2} - \sqrt{2\rho\varepsilon - \varepsilon^2} \\ &\leq (2\rho\delta + \delta^2 - \varepsilon^2 + \varepsilon^2) \cdot \frac{1}{2\sqrt{2\rho^2\varepsilon - \varepsilon^2}}. \end{aligned}$$

Since  $\delta \leq \varepsilon \leq 1 \leq \rho$ ,  $\rho\varepsilon \geq \varepsilon^2$  and  $\rho\delta \geq \delta^2$ . As a consequence,

$$\mathcal{T}\mathbf{1}_F(r) \lesssim \frac{\rho\delta}{\sqrt{\rho\varepsilon}}.$$

Thus

$$\begin{aligned}\|\mathcal{T}\mathbf{1}_F\|_q^q &\lesssim (\delta\rho)^{q/2}\rho^{d-2}\varepsilon + \left(\sqrt{\rho\varepsilon}\frac{\delta}{\varepsilon}\right)^q \rho^{d-1} \\ &\lesssim (\rho^{2-d})^{q/2-1}\varepsilon + \left(\sqrt{\rho}\frac{\rho^{1-d}}{\sqrt{\varepsilon}}\right)^q \rho^{d-1}.\end{aligned}$$

Now it is time to precise the value of  $\varepsilon$ . We recall that  $d \geq 3$  – the case  $d = 2$  was thoroughly studied by Christ, as noted earlier. As a consequence,  $q \geq 2$ . Thus

$$\|\mathcal{T}\mathbf{1}_F\|_q^q \lesssim \varepsilon + (\rho^{3/2-d}\varepsilon^{-1/2})^q \rho^{d-1}.$$

Now let us chose  $\varepsilon \sim 1/\rho$ . This leads to

$$\|\mathcal{T}\mathbf{1}_F\|_q^q \lesssim 1/\rho + \rho^{(1-d)(q-1)} \lesssim 1/R.$$

This proves the first step: the lemma is true when  $F$  is an interval.

Let us consider now a set  $F = E \cup I$  where  $I = (a, b)$  is an interval,  $a > \sup E$  and  $\Delta := d(E, I) > 0$ . Let us call now  $F_\Delta = E \cup I_\Delta$  where  $I_\Delta$  is an interval with  $\inf I_\Delta = a - \Delta$  and  $|I| = |I_\Delta|$  (we simply concentrate  $F$  by sticking its two parts to the closest one from 0). We want to show that

$$\|\mathcal{T}\mathbf{1}_{F_\Delta}\|_q^q \geq \|\mathcal{T}\mathbf{1}_F\|_q^q. \quad (2.1)$$

This is essentially an inverse concentration result. Indeed, it expresses the idea that the nearer from 0 a function is, the bigger its 1-plane transform is.

To prove (2.1), developing both sides, it would be sufficient to prove that for all  $1 \leq m \leq q - 1$  we have

$$\langle (\mathcal{T}\mathbf{1}_E)^{q-m}, (\mathcal{T}\mathbf{1}_{I_\Delta})^m - (\mathcal{T}\mathbf{1}_I)^m \rangle \geq 0.$$

Since  $\text{supp}(\mathcal{T}\mathbf{1}_E) \leq \sup E$ , we just have to prove that for all  $r \in E$ ,

$$\mathcal{T}\mathbf{1}_{I_\Delta}(r) - \mathcal{T}\mathbf{1}_I(r) \geq 0.$$

But if  $r \leq a$  then with the change of variable  $u^2 = v$ ,

$$\mathcal{T}\mathbf{1}_{I_\Delta}(r) - \mathcal{T}\mathbf{1}_I(r) = \int_{(a-\Delta)^2}^{a^2} \frac{dv}{2\sqrt{v-r^2}} - \int_{(b-\Delta')^2}^{b^2} \frac{dv}{2\sqrt{v-r^2}}. \quad (2.2)$$

Here  $\Delta'$  is such that  $I_\Delta = (a - \Delta, b - \Delta')$ . Because of  $|I_\Delta| = |I|$  this implies  $\Delta' \leq \Delta$ . The integrated function in (2.2) is nonincreasing (in  $u$ ) and because of  $\Delta' \leq \Delta$  we have  $\mathcal{T}\mathbf{1}_{I_\Delta}(r) \geq \mathcal{T}\mathbf{1}_I(r)$  for  $r \in \text{supp}(\mathcal{T}\mathbf{1}_E)$ .

Now let us consider a countable union of disjoint intervals,

$$F = \bigcup_{m \geq 0} I_m.$$

By the process described above, we can simply stick an arbitrary large, finite number of the intervals  $I_m$  to the closest one from 0. This increases the  $L^q$ -norm of the 1-plane transform. To pass from a finite number to an infinite number of  $I_m$  we simply notice

that  $|I| < \infty$ . Thus the lemma is true again.

Finally, assume that  $F$  is just a measurable set. Then by definition,

$$|F| = \inf \left\{ \sum_{m=1}^{\infty} |I_m|, I_m \text{ open interval with } F \subset \bigcup_{m \geq 0} I_m \right\}.$$

Approaching  $F$  with a recovering of intervals such that  $|\bigcup_{m \geq 0} I_m - F| \ll 1$  shows that the lemma remains true for any measurable set.  $\square$

Now using Lemma 2.2 we get

$$\|\mathcal{T}h_n\|_q \geq \|\mathcal{T}g_n\|_q - 2\delta_n R^{-d/p}. \quad (2.3)$$

Let us recall that  $B$  denotes the best constant in (1.2). Equation (2.3) leads to

$$B \geq \frac{\|\mathcal{T}h_n\|_q}{\|h_n\|_p} \geq \frac{\|\mathcal{T}g_n\|_q - 2\delta_n R^{-d/p}}{(1 - \delta_n)^{\frac{1}{p}}}.$$

Let us call  $\liminf_{n \rightarrow \infty} \delta_n(R) = l(R)$ . Then  $l(R) \leq 1$  and making  $n \rightarrow \infty$ ,

$$B \geq \frac{B - 2R^{-d/p}}{(1 - l(R))^{1/p}}.$$

This forces  $\limsup_{R \rightarrow \infty} l(R) = 0$ . In particular, there exists  $R_0$  and a subsequence of  $\mathcal{E}_n$ , still called  $\mathcal{E}_n$ , such that

$$|\mathcal{E}_n - [0, R_0]| \leq \frac{\epsilon}{2}.$$

Thus we call  $E_n = \mathcal{E}_n \cap [0, R_0]$  and Lemma 2.1 is proved. The proof in the case  $k = 1$  is similar, using Lemma 2.3.  $\square$

An easy consequence is the following:

**Corollary 3.** *Let  $f_n$  be an extremizing sequence. Then modulo the group of dilations,  $f_n$  admits a subsequence that converges weakly to a non-zero function in  $L^p$ .*

*Proof.* Let us consider  $g_n$ ,  $R_0$ ,  $E_n$  given by Lemma 2.1.  $g_n$  being bounded in  $L^p$ , it admits a subsequence that converges weakly to a function  $g$ . And  $g \neq 0$ :

$$\int_0^{R_0} g = \lim_{n \rightarrow \infty} \int_0^{R_0} g_n \geq \lim_{n \rightarrow \infty} \int_0^{R_0} \mathbf{1}_{E_n} \sim 1.$$

$\square$

Another consequence, related to the proof of Lemma 2.1 and Theorem 1, is the following:



**Corollary 4.** *The sequence  $g_n$  introduced in Lemma 2.1 admits a subsequence that is uniformly  $L^p$ -integrable, that is*

$$\lim_{R \rightarrow \infty} \int_{\{g_n > R\}} |g_n|^p = 0,$$

*uniformly in  $n$ .*

*Proof.* We recall that  $g_n$  was constructed such that the sequence  $g_n^*$  converged weakly to a non-zero function – see [5]. Moreover, the sequence  $g_n^*$  admits a subsequence that converges strongly – Theorem 1, part (i). Thus  $g_n^*$  admits a subsequence that is uniformly  $L^p$ -integrable, and so does  $g_n$ , using

$$\int_{\{g_n > R\}} |g_n|^p = \int_{\{g_n^* > R\}} |g_n^*|^p.$$

□

Note that here part (i) of Theorem 1 was really useful. Moreover, this approach can be generalized to any extremizing sequence that converges weakly to a nonzero function. This is a good starting point if one wants to generalize Christ's theorem in [3] for the Radon transform ( $k = d - 1$ ). Indeed, it excludes any form of Dirac-type concentration of the compactness. What we are missing is a complete quasiaxtremal theory for the inequality (1.1) such as the theorem 1.2 stated in [2] for  $k = d - 1$ , that helps rescaling extremizing sequences in a reasonable way.

From now we will consider  $g_n$  instead of  $f_n$ , and we will just assume the three following property, that are actually consequences of the above lemma and corollaries:

- The sequence  $g_n$  converges weakly to a non-zero function  $g \in L^p$ .
- There exist some sets  $E_n$  of measure  $|E_n| \sim 1$  such that  $g_n \geq 1_{E_n}$  and  $E_n \subset [0, R_0]$ .
- The sequence  $g_n$  is uniformly  $L^p$ -integrable.

### 3. WEAK INTERACTION

Let us recall the following famous lemma from Lions [8]:

**Lemma 3.1.** *Let  $\phi_n$  a sequence of nonnegative functions in  $L^1(\mathbb{R}^d)$ , such that  $\|\phi_n\|_1 = \lambda$ . Then there exists a subsequence of  $\phi_n$ , still noted  $\phi_n$ , such that one of the following is satisfied:*

- (1) (*tightness*) *There exists  $y_n \in \mathbb{R}^d$ , such that for all  $\varepsilon > 0$ , there exists  $R > 0$ , for all  $n$ ,*

$$\int_{B(y_n, R)} \phi_n \geq \lambda - \varepsilon.$$

(2) (*vanishing*) For all  $R$ ,

$$\lim_{n \rightarrow \infty} \sup_{y \in \mathbb{R}^d} \int_{B(y, R)} \phi_n = 0.$$

(3) (*dichotomy*) There exist  $0 < \alpha < \lambda$  and two sequences  $\phi_n^1, \phi_n^2$  of  $L^1$ -functions with compact support such that  $\phi_n \geq \phi_n^1, \phi_n^2 \geq 0$  and

$$\begin{aligned} \|\phi_n - \phi_n^1 - \phi_n^2\|_1 &\rightarrow 0, \quad \|\phi_n^1\|_1 \rightarrow \alpha, \quad \|\phi_n^2\|_1 \rightarrow \lambda - \alpha, \\ d(\text{supp}(\phi_n^1), \text{supp}(\phi_n^2)) &\rightarrow \infty. \end{aligned}$$

Let us consider the sequence  $\phi_n = |g_n|^p$ .  $\phi_n \in L^1(\mathbb{R}, r^{d-1}dr)$  and  $\|\phi_n\|_1 = 1$ , thus  $\phi_n$  satisfy one of the three consequences above. We can see, as a consequence of the previous section, that  $\phi_n$  does not vanish. Our purpose here is to prove that dichotomy cannot occur.

The first lemma that we want to show is a refinement of the dichotomy condition – in our particular case.

**Lemma 3.2.** *Let  $\phi_n = |g_n|^p$  be the sequence just above. Assume that the dichotomy condition (3) hold. Then up to a subsequence, there exist  $0 < \alpha < 1$ ,  $R'_0 > 0$ , and two sequences,  $\varphi_n^1, \varphi_n^2$ , of  $L^1$ -functions with compact support such that  $\phi_n \geq \varphi_n^1, \varphi_n^2 \geq 0$  and*

$$\|\phi_n - \varphi_n^1 - \varphi_n^2\|_1 \rightarrow 0, \quad \|\varphi_n^1\|_1 \rightarrow \alpha, \quad \|\varphi_n^2\|_1 \rightarrow 1 - \alpha,$$

*satisfying the additional support condition:*

$$\text{supp}(\varphi_n^1) \subset [0, R'_0], \quad \text{supp}(\varphi_n^2) \subset [R_n, \infty) \quad (3.1)$$

*with  $R_n \rightarrow \infty$ .*

This is a consequence of the fact that  $g_n \geq \mathbb{1}_{E_n}$  with  $|E_n| \sim 1$  and  $E_n \subset [0, R_0]$ , using measure theory and structure of open sets in  $\mathbb{R}$ .

*Proof.* Let us recall that there exist some sets  $E_n \subset [0, R_0]$  with measure of order 1 such that  $g_n \geq \mathbb{1}_{E_n}$ . Let us chose  $\varepsilon \leq |E_n|/2$ , and  $\phi_n^1, \phi_n^2$  associated by (iii) to this choice of  $\varepsilon$ . Then  $\phi_n^1$  or  $\phi_n^2$  must have some weight inside  $[0, R_0]$ , let's say  $\phi_n^1$ . The support separation property –  $d(\text{supp}(\phi_n^1), \text{supp}(\phi_n^2)) \rightarrow \infty$  – insures that  $\text{supp}(\phi_n^2) \cap [0, R_0] = \emptyset$  for  $n$  large enough.

Let us call  $K_n := \text{supp}(\phi_n^1)$ . There exists some open sets  $U_n$ , containing  $K_n$ , with  $|U_n - K_n| \ll 1$ . The structure of open sets in  $[0, \infty)$  allows us to write a decomposition

$$U_n = \bigcup_{i \geq 0} U_n^i$$

with  $U_n^i$  open, disjoint intervals, ordered by  $\sup U_n^i \leq \inf U_n^{i+1}$ , possibly enlarging  $U_n$  by an arbitrary small open set containing 0. Let us now call

$$d_n^i = d([0, R_0], U_n^i).$$

For all  $n$ , the sequence  $(d_n^i)_{i \geq 0}$  is increasing, and  $d_n^0 = 0$ . Using Cantor's diagonal argument, we can assume that for all  $i$ ,  $d_n^i \rightarrow d^i \in [0, \infty]$  as  $n \rightarrow \infty$ . Let us call  $i_0$  the smallest integer such that  $d^{i_0} = \infty$ . Note that since  $d_n^0 = 0$ , because of the existence of  $E_n$ , we have  $i_0 \geq 1$ . If  $i_0 = \infty$ , then the lemma is proved, we do not have to change  $\phi_n^1, \phi_n^2$ . If  $i_0 < \infty$ , let us call

$$U'_n := \bigcup_{i=0}^{i_0-1} U_n^i, \quad \rho_0 := \sup_{n \geq 0} \sup(U'_n) < \infty.$$

We can change the sequences  $\phi_n^1, \phi_n^2$  to

$$g_n^1 = \phi_n^1 \mathbf{1}_{U'_n}, \quad g_n^2 = \phi_n^1 \mathbf{1}_{U_n - U'_n} + \phi_n^2.$$

It remains to prove that  $g_n^1, g_n^2$  satisfy the conclusions of the lemma. Up to a subsequence, we can assume that  $\|g_n^1\|_1$  converges to some  $1 \geq \alpha' \geq 0$ . Moreover, we have  $0 \leq g_n^1 \leq \phi_n^1$  and thus  $\alpha' < 1$ . Since  $g_n^1 + g_n^2 = \phi_n^1 + \phi_n^2$ ,  $\|\phi_n - g_n^1 - g_n^2\|_1 \rightarrow 0$  and the additional support condition (3.1) is satisfied – with  $R_n = d_n^{i_0}$ . At last,  $\alpha' > 0$ : indeed,

$$\|\phi_n - g_n^1 - g_n^2\|_1 \rightarrow 0 \Rightarrow \int_0^{\rho_0} |\phi_n - g_n^1 - g_n^2| = \int_0^{\rho_0} |\phi_n - g_n^1| \rightarrow 0.$$

But  $\phi_n \geq \mathbf{1}_{E_n}$  with  $|E_n| \sim 1$  which implies  $\|g_n^1\|_1 \sim 1$ , thus  $\alpha' > 0$ .  $\square$

We will not change the notations for purpose of simplicity: we will assume without loss of generality  $\alpha' = \alpha$  and  $\rho_0 = R_0$ .

Now we are ready to prove our claim: dichotomy cannot occur. Let us be guided by the interpretation of Lemma 2.2 (or Lemma 2.3 in the case  $k = 1$ ). Those lemma express the fact that radial characteristic functions of sets that are far away from 0 cannot have a large  $k$ -plane transform. Here,  $g_n^2$  is far away from 0. However, we have no assumption on the support of the size of the support of  $g_n^2$  and this is a real difficulty: Lemma 2.2, 2.3 cannot be simply applied and have to be generalized to *any* function. Let us give an idea of the difficulty: roughly, let us write

$$g_n^2 = \sum_{m \in \mathbb{Z}} 2^m \mathbf{1}_{E_m}$$

with  $E_m = \{r \geq 0 \text{ with } g_n^2(r) \sim 2^m\}$ . Note that each  $E_m$  has to be far from 0. As a consequence,

$$\|\mathcal{T} g_n^2\|_q^q = \sum_{m \in \mathbb{Z}} 2^{mq} \|\mathcal{T} \mathbf{1}_{E_m}\|_q^q + \sum_{m_1, \dots, m_q} 2^{(m_1 + \dots + m_q)q} \int \mathcal{T} \mathbf{1}_{E_{m_1}} \cdot \dots \cdot \mathcal{T} \mathbf{1}_{E_{m_q}}.$$

The first sum is called principal sum (and its components principal terms) while the second is called interaction sum (and its components interaction terms); this is a notion we will encounter again below. It would not be too hard to prove that the principal sum tends to 0 as  $n$  tends to infinity. This would follow from Lemma 2.2. However, there are too many terms in the interaction sum. Lemma 2.2 would not be of any help. Approaches of this type have however been made in [2], see the proof of Theorem 1.6.

We believe it is possible to treat it with the same methods. However, we have chosen to take another, simpler, path.

Indeed let us get inspired by Lions in his famous paper [8]. We introduce the quantities

$$S_\alpha := \sup\{\|\mathcal{T}f\|_q^q, \|f\|_p^p = \alpha\}.$$

Then the following convexity inequality is satisfied: for all  $0 < \alpha < 1$ ,

$$S_1 > S_\alpha + S_{1-\alpha}. \quad (3.2)$$

Indeed, we have the relation  $S_\alpha = \alpha^{\frac{q}{p}} S_1$ . For all  $\alpha \in (0, 1)$  we have

$$1 > \alpha^{\frac{q}{p}} + (1 - \alpha)^{\frac{q}{p}},$$

recalling  $q > p$ . Thus (3.2) holds.

Assume now that dichotomy can occur. Then because of Lemma 3.2 there exist two sequences  $g_n^1, g_n^2$ , satisfying

- (i)  $\text{supp}(g_n^1) \subset [0, R_0]$ ,  $d(\text{supp}(g_n^1), \text{supp}(g_n^2)) \rightarrow \infty$ .
- (ii)  $\|g_n^1\|_p^p \rightarrow \alpha \in (0, 1)$ ,  $\|g_n^2\|_p^p \rightarrow 1 - \alpha$  as  $n \rightarrow \infty$ .
- (iii)  $\||g_n|^p - |g_n^1 + g_n^2|^p\|_1 \rightarrow 0$  as  $n \rightarrow \infty$ .
- (iv)  $g_n \geq g_n^1, g_n^2 \geq 0$ .

Our purpose is to prove a contradiction with (3.2), using Lemma 3.4. The essential idea is the following lemma:

**Lemma 3.3.** *Let  $g_n, g_n^1, g_n^2$  as above. Then*

$$\|\mathcal{T}g_n\|_q^q - \|\mathcal{T}g_n^1\|_q^q - \|\mathcal{T}g_n^2\|_q^q \rightarrow 0. \quad (3.3)$$

It means in a way that  $\mathcal{T}g_n^1$  and  $\mathcal{T}g_n^2$  interact weakly, or are asymptotically orthogonal. This will be a contradiction with (3.2). It is interesting to relate this lemma to the discussion made above. Here we do not make  $\mathcal{T}g_n^2$  interact with itself, but with  $\mathcal{T}g_n^1$ . Since the supports of  $g_n^1$  and  $g_n^2$  are far away from each other it is actually much easier.

*Proof.* Let us first define  $\varepsilon_n := g_n - g_n^1 - g_n^2$ . Then  $\|\varepsilon_n\|_p \rightarrow 0$ . Indeed,

$$\begin{aligned} \|\varepsilon_n\|_p^p &= \int_0^\infty |g_n - g_n^1 - g_n^2|^p \\ &\leq \int_0^\infty |g_n|^p - |g_n^1 + g_n^2|^p \rightarrow 0, \end{aligned}$$

because of (iii) in Lemma 3.1. Moreover,

$$\begin{aligned} \|\mathcal{T}g_n\|_q^q &\leq \|\mathcal{T}g_n^1 + \mathcal{T}g_n^2\|_q^q + O(\|\varepsilon_n\|_p) \\ &\leq \|\mathcal{T}g_n^1\|_q^q + \|\mathcal{T}g_n^2\|_q^q + \sum_{1 \leq m \leq q-1} \binom{q}{m} \langle (\mathcal{T}g_n^1)^{q-m}, (\mathcal{T}g_n^2)^m \rangle + o(1). \end{aligned}$$

To prove (3.3) it is sufficient to show that for  $1 \leq m \leq q-1$ ,  $\langle (\mathcal{T}g_n^1)^{q-m}, (\mathcal{T}g_n^2)^m \rangle \rightarrow 0$ . Let  $\varepsilon > 0$ . Since  $g_n$  is uniformly  $L^p$ -integrable, there exists  $R_\varepsilon$  such that

$$\int_{g_n^1 \geq R_\varepsilon} |g_n^1|^p \leq \int_{g_n \geq R_\varepsilon} |g_n|^p \leq \varepsilon,$$

uniformly in  $n$ . Thus, in a way, it is sufficient to prove (3.3) for  $g_n^1 \leq R_\varepsilon$ . Indeed,

$$\begin{aligned} & |\langle (\mathcal{T}g_n^1)^{q-m}, (\mathcal{T}g_n^2)^m \rangle - \langle (\mathcal{T}\mathbb{1}_{\{g_n^1 \leq R_\varepsilon\}}g_n^1)^{q-m}, (\mathcal{T}g_n^2)^m \rangle| \\ &= |\langle (\mathcal{T}(\mathbb{1}_{\{g_n^1 \leq R_\varepsilon\}}g_n^1 + \mathbb{1}_{\{g_n^1 > R_\varepsilon\}}g_n^1))^{q-m} - (\mathcal{T}\mathbb{1}_{\{g_n^1 \leq R_\varepsilon\}}g_n^1)^{q-m}, (\mathcal{T}g_n^2)^m \rangle| \\ &\leq \sum_{1 \leq i \leq m-q} \binom{m-q}{i} |\langle (\mathcal{T}(\mathbb{1}_{\{g_n^1 \leq R_\varepsilon\}}g_n^1)^{q-m-i} (\mathcal{T}\mathbb{1}_{\{g_n^1 > R_\varepsilon\}}g_n^1)^i), (\mathcal{T}g_n^2)^m \rangle|. \end{aligned}$$

Let us treat independently the quantities  $\langle (\mathcal{T}(\mathbb{1}_{\{g_n^1 \leq R_\varepsilon\}}g_n^1)^{q-m-i} (\mathcal{T}\mathbb{1}_{\{g_n^1 > R_\varepsilon\}}g_n^1)^i), (\mathcal{T}g_n^2)^m \rangle$  that appeared just above. Using Hölder's inequality with

$$1 = \frac{q-m-i}{q} + \frac{i}{q} + \frac{m}{q},$$

we get

$$\begin{aligned} & |\langle (\mathcal{T}(\mathbb{1}_{\{g_n^1 \leq R_\varepsilon\}}g_n^1)^{q-m-i} (\mathcal{T}\mathbb{1}_{\{g_n^1 > R_\varepsilon\}}g_n^1)^i), (\mathcal{T}g_n^2)^m \rangle| \\ &\leq \|(\mathcal{T}\mathbb{1}_{\{g_n^1 \leq R_\varepsilon\}}g_n^1)\|_q^{q-m-i} \cdot \|\mathcal{T}\mathbb{1}_{\{g_n^1 > R_\varepsilon\}}g_n^1\|_q^i \cdot \|\mathcal{T}g_n^2\|_q^m \\ &\leq A(k, d)^q \cdot \|g_n^1\|_p^{q-m-i} \cdot \|\mathbb{1}_{\{g_n^1 > R_\varepsilon\}}g_n^1\|_p^i \cdot \|g_n^2\|_p^m \\ &\lesssim \varepsilon^{\frac{i}{p}}. \end{aligned}$$

Coming back to our initial point,

$$|\langle (\mathcal{T}g_n^1)^{q-m}, (\mathcal{T}g_n^2)^m \rangle - \langle (\mathcal{T}\mathbb{1}_{\{g_n^1 \leq R_\varepsilon\}}g_n^1)^{q-m}, (\mathcal{T}g_n^2)^m \rangle| \lesssim \varepsilon^{1/p}$$

and this bound is uniform in  $n$ . Thus we can assume without loss of generality  $g_n^1 \leq R_\varepsilon$ . This leads to

$$\begin{aligned} \langle (\mathcal{T}g_n^1)^{q-m}, (\mathcal{T}g_n^2)^m \rangle &\leq R_\varepsilon^{q-m} \langle (\mathcal{T}\mathbb{1}_{[0, R_0]})^{q-m}, (\mathcal{T}g_n^2)^m \rangle \\ &\lesssim R_\varepsilon^{q-m} R_0^{(q-m)k} \langle \mathbb{1}_{[0, R_0]}, (\mathcal{T}g_n^2)^m \rangle. \end{aligned}$$

**Lemma 3.4.** *Let  $\psi \in L^p$  and  $R \geq 1$  such that  $\delta := d(\text{supp}(\psi), [0, R]) \geq R$ . Then for all  $1 \leq m \leq q-1$ ,*

$$\langle \mathbb{1}_{[0, R]}, (\mathcal{T}\psi)^m \rangle \lesssim \frac{R^{d-k}}{(R + \delta)^{\frac{m}{p}}} \|\psi\|_p^m.$$

*Proof.* It is only some simple calculation. Indeed,

$$\begin{aligned}
\langle \mathbb{1}_{[0,R]}, (\mathcal{T}\psi)^m \rangle &= \int_0^R \left( \int_{u \geq r} \psi(u) (u^2 - r^2)^{k/2-1} u du \right)^m r^{d-k-1} dr \\
&= \int_0^R \left( \int_{u_1 \geq R+\delta} \psi(u_1) k(u_1, r) u_1 du_1 \right) \cdot \dots \cdot \left( \int_{u_m \geq R+\delta} \psi(u_m) k(u_m, r) u_m du_m \right) r^{d-k-1} dr \\
&= \int_{u_1 \geq R+\delta} \dots \int_{u_m \geq R+\delta} \psi(u_1) u_1 du_1 \dots \psi(u_m) u_m du_m \int_0^R k(u_1, r) \dots k(u_m, r) r^{d-k-1} dr,
\end{aligned}$$

where we note  $k(u, r) = (u^2 - r^2)^{k/2-1}$ . For  $k \geq 2$ ,

$$\int_0^R k(u_1, r) \dots k(u_m, r) r^{d-k-1} dr \leq u_1^{k-2} \dots u_m^{k-2} R^{d-k}.$$

For  $k = 1$ ,

$$\begin{aligned}
\int_0^R k(u_1, r) \dots k(u_m, r) r^{d-k-1} dr &\leq \int_0^R \frac{1}{\sqrt{u_1^2 - r^2}} \dots \frac{1}{\sqrt{u_m^2 - r^2}} r^{d-k-1} dr \\
&\leq \frac{1}{u_1} \dots \frac{1}{u_m} \int_0^R \frac{1}{\sqrt{1 - \frac{r^2}{u_1^2}}} \dots \frac{1}{\sqrt{1 - \frac{r^2}{u_m^2}}} r^{d-k-1} dr \\
&\lesssim \frac{1}{u_1} \dots \frac{1}{u_m} R^{d-k}
\end{aligned}$$

since  $u_i \geq R + \delta \geq 2R \geq 2r$ .

As a consequence,

$$\begin{aligned}
\langle \mathbb{1}_{[0,R]}, (\mathcal{T}\psi)^m \rangle &\lesssim \int_{u_1 \geq R+\delta} \dots \int_{u_m \geq R+\delta} \psi(u_1) u_1 du_1 \dots \psi(u_m) u_m du_m u_1^{k-2} \dots u_m^{k-2} R^{d-k} \\
&= R^{d-k} \left( \int_{u \geq R+\delta} \psi(u) u^{k-1} du \right)^m.
\end{aligned}$$

Noting that

$$u^{k-1} = u^{-\frac{2}{p'}} u^{\frac{d-1}{p}},$$

the Hölder inequality leads to:

$$\int_{u \geq R+\delta} \psi(u) u^{k-1} du \lesssim \left( \int_{u \geq R+\delta} \psi(u)^p u^{d-1} du \right)^{1/p} \left( \int_{u \geq R+\delta} u^{-2} du \right)^{\frac{1}{p'}} = \frac{1}{(R+\delta)^{\frac{1}{p'}}} \|\psi\|_p.$$

Thus, finally,

$$\langle \mathbb{1}_{[0,R]}, (\mathcal{T}\psi)^m \rangle \lesssim \frac{R^{d-k}}{(R+\delta)^{\frac{m}{p'}}} \|\psi\|_p^m.$$

□

For  $n$  large enough, a direct application of Lemma 3.4 leads to

$$\langle \mathbb{1}_{[0, R_0]}, (\mathcal{T}g_n^2)^m \rangle \lesssim \frac{R_0^{d-k}}{(R_0 + d(R_0, \text{supp}(g_n^2)))^{\frac{m}{p'}}} \|g_n^2\|_p^m \lesssim \frac{1}{d(R_0, \text{supp}(g_n^2))^{\frac{m}{p'}}}.$$

Thus we get

$$\langle (\mathcal{T}g_n^1)^{q-m}, (\mathcal{T}g_n^2)^m \rangle \lesssim R_\varepsilon^{q-m} R_0^{(q-m)k} \frac{R_0^{d-k}}{d(R_0, \text{supp}(g_n^2))^{\frac{m}{p'}}}.$$

Making  $n \rightarrow \infty$  leads to the conclusion – here we notice that the uniformity of  $R_\varepsilon$  in  $n$  is crucial.  $\square$

Now let us chose two sequences  $\alpha_n, \beta_n \rightarrow 1$  with  $\|\alpha_n g_n^1\|_p^p = 1$  and  $\|\beta_n g_n^2\|_p^p = 1 - \alpha$ . Then

$$S_\alpha + S_{1-\alpha} \leftarrow S_\alpha + S_{1-\alpha} + o(1) \geq \|\mathcal{T}\alpha_n g_n^1\|_q^q + \|\beta_n \mathcal{T}g_n^2\|_q^q + o(1) = \|\mathcal{T}g_n\|_q^q \rightarrow S_1$$

which is a contradiction with (3.2). Thus dichotomy cannot occur.

An easy consequence is the following:

**Corollary 5.**  $g_n$  is strongly tight, that is

$$\lim_{R \rightarrow \infty} \int_R^\infty |g_n|^p = 0,$$

uniformly in  $n$ .

Indeed, since  $|g_n|^p \geq \mathbb{1}_{E_n}$ ,  $g_n$  cannot be vanishing. Thus  $|g_n|^p$  has to be tight. But since  $E_n \subset [0, R_0]$  the sequence  $y_n$  involved in (i) in Lemma 3.1 can be chosen to be 0, involving a possible redefinition of  $R_0$ .

#### 4. STRONG CONVERGENCE TO AN EXTREMIZER

Here we prove our main result, Theorem 2. It is an easy consequence of the following theorem:

**Theorem 6.**  $\mathcal{T}g_n$  converges strongly in  $L^q$ .

We first start to prove that the operator  $\mathcal{T}$  is somehow locally compact.

**Lemma 4.1.** Let us consider for  $R > 0$  the operator

$$\begin{aligned} \mathcal{T}_R : L^\infty([0, R]) &\longrightarrow L^q([0, R]) \\ f &\longmapsto \mathcal{T}\mathbb{1}_{[0, R]}f = \mathcal{T}f. \end{aligned}$$

Then  $\mathcal{T}_R$  is compact.

We give the proof of this result in the appendix. Let us now define

$$\begin{aligned} g_n^m &:= \mathbb{1}_{\{f_n \leq m\}} \mathbb{1}_{[0, m]} g_n, \\ h^m &:= \lim_{n \rightarrow \infty} \mathcal{T}g_n^m. \end{aligned} \tag{4.1}$$

The convergence in (4.1) occurs in  $L^q$ , because of the local compactness of  $\mathcal{T}$  – Lemma 4.1. Indeed, the sequence  $(g_n^m)_n$  is bounded in  $L^\infty([0, m])$ , thus the sequence  $(\mathcal{T}g_n^m)_n$  is compact in  $L^q$ . Moreover  $0 \leq g_n^m \leq g_n^{m+1} \leq g_n$ , implying  $0 \leq \mathcal{T}g_n^m \leq \mathcal{T}g_n^{m+1} \leq \mathcal{T}g_n$ . It proves that the sequence  $h^m$  is actually nondecreasing, nonnegative, and bounded in  $L^q$ . We can then apply the monotonous convergence theorem: there exists  $h \in L^q$  with  $h^m \rightarrow h$  strongly. We want to show now that  $\mathcal{T}g_n$  converges strongly to  $h$ .

$$\lim_{n \rightarrow \infty} \mathcal{T}g_n - h = \lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} \mathcal{T}g_n - h^m \quad (4.2)$$

$$= \lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} \mathcal{T}g_n - \mathcal{T}g_n^m \quad (4.3)$$

$$= \lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} \mathcal{T}g_n - \mathcal{T}g_n^m = \lim_{n \rightarrow \infty} 0 = 0. \quad (4.4)$$

In (4.4), we were allowed to change the order of the limits  $n \rightarrow \infty$ ,  $m \rightarrow \infty$  because of the uniform convergence (in  $n$ ) of  $g_n^m$  to  $g_n$  as  $m \rightarrow \infty$ .

Now we know that  $\mathcal{T}$  is linear continuous from  $L^p$  to  $L^q$ , so it is also continuous when these spaces are provided with the weak topology. It shows that  $\mathcal{T}g_n \rightharpoonup \mathcal{T}g$ . But  $\mathcal{T}g_n \rightarrow h$ , thus  $\mathcal{T}g = h$  and  $\mathcal{T}g_n \rightarrow \mathcal{T}g$ . This proves Theorem 6.

It implies that the sequence  $g_n$  converges weakly to an extremizer, since  $\|g\|_p \leq \liminf \|g_n\|_p = 1$ . This also implies  $\|g\|_p = 1$ . Let us introduce some general functional analysis that explains why this proves Theorem 2.

We say that a Banach space  $E$  is uniformly convex when for all  $\varepsilon > 0$ , there exists  $\delta > 0$  such that for all  $x, y$  lying in the unit ball of  $E$ , such that  $\|x - y\|_E \geq \varepsilon$ , we have

$$\frac{\|x + y\|_E}{2} \leq 1 - \delta.$$

For instance, it is a standard result that  $L^p$  is a uniformly convex space for all  $1 < p < \infty$ . Uniformly convex space have the following useful property, that has been somehow pointed to us by Christ:

**Lemma 4.2.** *Let  $E$  a uniformly convex Banach space and  $x_n$  a sequence in  $E$  such that  $x_n \rightharpoonup x \in E$ . Assume moreover that  $\lim \|x_n\|_E = \|x\|_E$ . Then  $x_n \rightarrow x$  in  $E$ .*

*Proof.* If  $\|x\| = 0$ , it is obvious. Then we can assume  $x \neq 0$  and call  $y_n := x_n / \|x_n\|_E$ ,  $y = x / \|x\|_E$ . Then we have

$$\frac{y + y_n}{2} \rightharpoonup y$$

which implies

$$1 = \|y\|_E \leq \liminf_{n \rightarrow \infty} \frac{\|y_n + y\|_E}{2} \leq 1.$$

Because of the uniform convexity of  $E$ ,  $x_n \rightarrow x$ . □

Since  $1 < p < \infty$ , a direct application of this lemma gives  $g_n \rightarrow g$  and thus Theorem 2 is proved.



APPENDIX 1: COMPACTNESS OF THE RESTRICTED OPERATOR  $\mathcal{T}_R$ 

Here we want to prove lemma 4.1. The operator  $\mathcal{T}_R$  is formally defined as  $\mathcal{T}_R := \mathcal{T}\mathbb{1}_{[0,R]}$  and maps  $L^\infty([0, R])$  to itself. Moreover, since  $L^\infty([0, R]) \hookrightarrow L^q([0, R])$ , we just have to prove that  $\mathcal{T}_R : L^\infty([0, R]) \rightarrow L^\infty([0, R])$  is compact.

Compactness in  $L^\infty([0, R])$  is expressed through the standard Arzelà-Ascoli theorem:

**Theorem 7.** *Let  $\mathcal{P} \subset L^\infty([0, R])$ . Then  $\mathcal{P}$  is relatively compact if and only if it is bounded in  $L^\infty([0, R])$  and equicontinuous.*

Thus we want to show that  $\mathcal{P} := \mathcal{T}(\{f \in L^\infty([0, R], \|f\|_\infty \leq 1\})$  is a equicontinuous family of functions in  $L^\infty$ . Let  $f \in \mathcal{P}$ . Then for all  $0 \leq r \leq r+h \leq R$ ,

$$\begin{aligned} |\mathcal{T}f(r+h) - \mathcal{T}f(r)| &\leq \int_0^R |f(u)| |\mathbb{1}_{u \geq r+h}(u^2 - (r+h)^2)^{k/2-1} - \mathbb{1}_{u \geq r}(u^2 - r^2)^{k/2-1}| u du \\ &\leq \|f\|_\infty \int_0^R |\mathbb{1}_{u \geq r+h}(u^2 - (r+h)^2)^{k/2-1} - \mathbb{1}_{u \geq r}(u^2 - r^2)^{k/2-1}| u du \\ &:= \|f\|_\infty I(h, r). \end{aligned}$$

We just want to estimate  $I(h, r)$ .

$$\begin{aligned} I(h, r) &\leq \int_{r+h}^R |(u^2 - (r+h)^2)^{k/2-1} - (u^2 - r^2)^{k/2-1}| u du \\ &\quad + \int_0^R |\mathbb{1}_{u \geq r+h} - \mathbb{1}_{u \geq r}| (u^2 - r^2)^{k/2-1} u du. \end{aligned}$$

With the change of variable  $u^2 = v$ , this gives

$$\begin{aligned} I(h, r) &\lesssim \int_{(r+h)^2}^{R^2} |(v - (r+h)^2)^{k/2-1} - (v - r^2)^{k/2-1}| dv \\ &\quad + \int_0^{R^2} |\mathbb{1}_{v \geq (r+h)^2} - \mathbb{1}_{v \geq r^2}| (v - r^2)^{k/2-1} dv \\ &\lesssim \int_{(r+h)^2}^{R^2} |(v - (r+h)^2)^{k/2-1} - (v - r^2)^{k/2-1}| dv \\ &\quad + \int_{r^2}^{(r+h)^2} (v - r^2)^{k/2-1} dv. \end{aligned}$$

By Holder's inequality,

$$\int_{r^2}^{(r+h)^2} (v - r^2)^{k/2-1} dv \leq h^{1/3} \left( \int_{r^2}^{R^2} ((v - r^2)^{k/2-1})^{3/2} dv \right)^{2/3}$$

and this converges to 0 as  $h$  converges to 0. An estimate for the other term is

$$\begin{aligned}
\int_{(r+h)^2}^{R^2} |(v - (r+h)^2)^{k/2-1} - (v - r^2)^{k/2-1}| dv &\leq \int_{(r+h)^2}^{R^2} \int_{r^2}^{(r+h)^2} \left| \frac{k}{2} - 1 \right| s |v - s|^{k/2-2} ds dv \\
&\lesssim \int_{r^2}^{(r+h)^2} \int_{(r+h)^2}^{R^2} s |v - s|^{k/2-2} dv ds \\
&\lesssim \int_{r^2}^{(r+h)^2} s ((R^2 - s)^{k/2-1} - ((r+h)^2 - s)^{k/2-1}) ds.
\end{aligned}$$

Here again, an application of Holder's inequality makes us conclude. Thus  $\mathcal{P}$  is equicontinuous and  $\mathcal{T}_R$  is compact.

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